**Spontaneous Decay**

One finds that excited states of the H-atom are not actually stable, even in vacuum. The explanation behind this is that vacuum fluctuations of the photon field can induce a transition between states. Basically, eigenstates of Hatom are not eigenstates of Hatom + HEM field, and so are not truly stationary states. Taking a gander at the next file, our H for the Hydrogen atom – EM field system would be, in Natural Gaussian units (and using QFT phase convention on a, a†):



(**p**·**A** + **A**·**p** = 2**A**·**p** because **p** can slide past **A** since in Coulomb gauge ∇·**A** = 0) where go out to just first order in A in the last line, and where in faux-Gaussian units, in the continuum limit, we have:



But we’ll set t = 0, because we’re in the Schrodinger picture here, and we’ll set **r** = 0 because hopefully approximating the vector field at the origin is good enough – certainly should be for light wavelengths long in comparison to the size of the atom but whatever. Suppose we would start with an e- in a 2ℓm eigenstate of the atom. We would like to calculate the rate of transition to the ground state (which is essentially the inverse lifetime of the state). We’ll actually look to calculate the rate of transition:



where |0> is the GS of the photon field (we’ll assume that we’re at T = 0 basically), and



(see boson file) So we need to calculate, using 1st order perturbation theory (and leaving off operator hats), and note that k in the δ function is the energy of the photon in our units, which would be ℏkc = hf, if we were to go back to SI.



(Γni is transition rate from |i> = |2ℓm>|0> to |n> = |100>|kλ>) To evaluate the matrix element, we use following trick,



So,



which is easier…so now we need to evaluate:



where I’m replacing ε100 – ε210 w/ ωk, thanks to the delta function. Now consider <100||2ℓm>. Recall that is a 1st rank spherical tensor operator, and also odd under parity (see QM Foundations/Tensor Operators & Parity Symmetry). So the only non-zero expectations that we’ll get from this require:



(maybe see QM/Time-Dependent/RSPT Scattering Perturbation too) This means that ℓ will have to be 1. So now we can specialize to <100||21m>, but m can be anything. Let’s work this out I guess. Looking back at the QM/Time-Independent folder/Hydrogenic atom file, we have:



where,



And so, setting Z = 1, we have:



and,



So the expectation is:



To facilitate working this out, it’s convenient to put the vector **r** in terms of spherical harmonics. Consider Y1m(θ,φ). We have,



So we have:



Filling this in,



Now use fact that Y00 = 1/√(4π). And also guess I’ll need to use (see QM Foundations/Time Reversal Symmetry file):



So,



Still simplifying,



Now going back, again, to:



Let’s work it out,



Expanding the sum,



Let’s average over m via (1/3)Σm. Then some of the terms go to zero due to incompatibility of δ functions. And we have:



And now doing the sum,



So now we have:



Finally, we should sum over all values of **k**, which will give us the total scattering rate, and hence, the inverse lifetime. The ‘sum over **k**’ will take the form ∫d3k. This is appropriate since our |**k**λ> was normalized to <**k**λ|**k**´λ´> = δ(k-k´)δλλ´, as alluded to in our comment near the top of the page about how to define |**k**λ> w/r to the vacuum state. So,



Continuing on,



Now in our units 4πε­0 = 1, c = 1, ℏ = 1, we have:



So,



Okay, well we’re going to write this in terms of the fine structure constant. So in our units:



So now,



where α is the fine structure constant. This result is in the unphysical Natural + Gaussian units (see Units file), where mass is equivalent to inverse time. But we want to convert this to MKS units. So we need to restore factors of 4πε0, µ0/4π, and ℏ. So, as discussed in that units file, we’d require:



So we need:



So blue equation says p = q. Plugging this into red equation, we see r = -1. Back into green equation means p = -1. And therefore q = -1. So our answer would be in MKS,



Plugging in m = 9.11×10-31kg, c = 3×108m/s, h = 6.34×10-34Js, and α = 1/137 (about), we get:



Inverting, we have:



which matches experimental measurements.